an introduction to ALGEBRA

Paige Zhu
Dragon Curve Tutoring
dragoncurvetutoring.org

Preface

Welcome to the first book in our Competition Math series! We will try cover most of the important competition math concepts in this series, so be ready for some fun with the materials and problems! To start this book, make sure you have strong arithmetic skills and some basic algebra skills too.

This book will have useful formulas, strategies, practice problems, and lots of important concepts. If you are a seasoned problem solver, skim through the book and solve some of the tougher problems. If you are just beginning problem solving, thoroughly read each section and solve the problems. Try your best at solving the problems, before looking at the solutions.

This book is published under the non-profit organization Dragon Curve Tutoring. Our mission at Dragon Curve Tutoring is to make high-quality math education accessible to underserved communities, igniting a passion for learning and problem-solving. As you embark on this mathematical journey, know that this book is the result of collaboration, dedication, and a shared belief in the transformative power of education.

I want to extend my heartfelt gratitude to all contributors who have made this project possible. Aritra12, Aops-g5-gethsemanea2, Rusczyk, MeHateMemes, RohanQV, and mathbw225, your contributions have shaped this volume into a valuable resource for everyone.

This book is organized into three chapters: Algebra, Graphing Equations, and Complex Numbers. Each chapter builds upon the last, providing a comprehensive and structured approach to mastering algebraic concepts. Whether you're a budding math enthusiast or someone seeking to strengthen their problem-solving skills, the carefully curated content within these pages will guide you step by step.

Looking ahead, this book marks just the beginning of our series. We are currently developing three more books, each dedicated to a specific branch of mathematics: Geometry, Number Theory, and Combinatorics. Our belief is rooted in fostering a well-rounded foundation for learners, and we hope that these books will empower young minds to excel across a spectrum of mathematical domains.

With so much to remember, I want to extend my heartfelt thanks to you, our reader, for dedicating time to read this book. Your commitment to learning and personal growth is what propels Dragon Curve Tutoring's mission forward. I hope this volume inspires a lasting appreciation for mathematics and acts as a catalyst for your academic journey.

`Happy learning!`

Contents

1 Algebraic Equations

1.1 Introduction

Algebra is the study of **variables**. A variable is a letter in place of an unknown value. Let's say we have a mystery number. We can use a variable, say a, to represent the mystery number. It is easier to write "a" than "the mystery number". Variables are just place-holders for a number, so anything you can do with numbers, like addition and subtraction, you can do with variables.

In this chapter we explore different kinds of algebra problems and equations that appear in many math contests worldwide. These questions require you to try out different tactics and methods. Experiment and see if you can find patterns. If you discover something yourself, you'll remember it better.

1.2 Linear Equations

Questions with one variable only require simple arithmetic computations. In the questions below, try to solve for the variable. To do this, try to get the variable to one side and all other numeric values on the other side.

Exercises

Before we do some word problems, test your algebra skills by solving these equations with one variable.

1.2.1 $3x + \frac{1}{2} = \frac{43}{2}$

1.2.2 $(y - 5) \cdot (-1) + 2 = 3$

1.2.3 $12z - 66 = 5z$

1.2.4 $\frac{3+a}{2} = \frac{7+a}{3}$

1.2.5 $\frac{3+b}{2b-2} = \frac{5}{2}$

Now that you're warmed up, let us see how algebra can help in other kinds of problems.

Example 1.01

Janet picks her favorite number. She adds four and then halves the result. Finally, she subtracts six. She gets six and tells the result to her friend Jeremiah. What is Janet's favorite number?

SOLUTION Let's turn all these words into math. Suppose Janet's favorite number is j. Now, let's re-read the problem and follow the operations she did on her favorite number. First, Janet adds four. That means she got

$$j + 4.$$

She halves her answer and subtracts six. Then, she gets

$$\frac{(j+4)}{2} - 6.$$

After doing all these calculations, her answer is 6. But we just found out that her result is $\frac{(j+4)}{2} - 6$. So, we must have

$$\frac{j+4}{2} - 6 = 6.$$

Now we can solve the equation just as we did in the exercises above. Adding six to both sides gives us

$$\frac{j+4}{2} = 12.$$

Multiplying both sides by 2 to remove the fraction, we obtain

$$j + 4 = 24.$$

Finally, we subtract four from both sides to find that $j = 20$. So Janet's favorite number is $\boxed{20}$.

Let's look at another example of a word problem that can be turned into math.

Example 1.02

Keri bought 3 packs of gum at her local convenience store. Her friend Jessica bought 5 packs of gum. Each pack of gum has g pieces of gum. Find an expression in terms of x that represents the amount of gum Keri and Jessica have together.

SOLUTION If Keri bought 3 packs of gum that each have g pieces of gum, she has a total of $3 \cdot g$ pieces of gum. Similarly, if Jessica bought 5 packs of gum, she has a total of $5 \cdot g$ pieces of gum.

Thus, they have a total of

$$3g + 5g = 8g$$

An expression to represent the amount of gum Keri and Jessica have is $8g$.

What if there are two mystery numbers? Let us explore it in our last example:

Example 1.03

Marcello says to Aiden, "If you give me one of your pencils, I'll have twice as many pencils as you." Aiden answers, "But if you give me one of yours, we'll have the same number of pencils each." How many pencils does Aiden have? *(Source: Australian Mathematics Competition)*

SOLUTION So now we have two variables. Let's call m the number of pencils Marcello had originally and a for the number of pencils Aiden had originally. If Aiden gives Marcello one of his pencils, Marcello will have twice as many pencils as Aiden. In algebraic form:

$$m + 1 = 2(a - 1)$$

Also, if Marcello gives Aiden one of his, they will have the same number of pencils:

$$m - 1 = a + 1$$

From the equation directly above, we see that $a + 2 = m$. This gives us another clue to substitute m and find a directly into the first equation. After substituting we get:

$$a + 3 = 2(a - 1)$$

Expanding the right side, we get:

$$a + 3 = 2a - 2.$$

Finally, subtracting a from both sides, we have

$$3 = a - 2$$

and adding 2 on both sides yields

$$5 = a$$

So, Aiden has $\boxed{5}$ pencils.

Exercises

Attempt these word problems. Try to think in numbers and give names to the values that you don't know. Then try to make relations and equations to solve them.

1.2.6 A soccer team is made out of defenders, strikers, and midfielders. 19 players are defenders, 9 players are strikers, and $\frac{3}{7}$ of the whole team are midfielders. What is the total number of soccer players on the team? *(Source: SASMO)*

1.2.7 The number $\frac{3}{2}$ is equivalent to three more than twice a certain number. What is that number?

1.2.8 I am thinking of a number that when you multiply it by 3, add 3, then subtract 2, you get $\frac{17}{4}$. What is ten times the number?

1.2.9 If x is a number such that $2x + 5 = 9$, what is the value of the expression $(2x + 6)(2x + 7)$? *(Challenge: Can you think of a shortcut?)*

1.2.10 If $a + b = 12$ and $ab = 23$, then find $a^2 + b^2$.

Challenge Problems

Try the questions below. These are not necessarily linear algebra, but require your previous knowledge of math, linear algebra and some creativity. You might even be surprised by the simplicity of the method!

1.2.11 If $\sqrt{8}+\sqrt{18} = \sqrt{x}$, what is the value of x? *(Source: MATHCOUNTS)*

1.2.12 If a and b are positive integers such that $a^2 + b^2 = 197$, what is the value of $a+b$? *(Source: MATHCOUNTS)*

1.2.13$\star$ Let $m = 101^4 + 256$. Find the sum of the digits of m. *(Source: AIME)*

1.3 Quadratic Equations

Definition

A quadratic equation is an equation in which the highest power of an unknown quantity is 2 i.e. in the form of

$$ax^2 + bx + c = 0$$

, where a, b, and c are placeholders for (any real) constant number.

Here is an example of a quadratic equation: $x^2 - 2x + 1 = 0$. According to the definition, we can determine the constants a, b and c of the above equation, which is 1, -2, and 1 respectively. Compare the quadratic equation example with the definition of a quadratic.

Other examples:

- $x^2 = 0$ (a=1, b=0, c=0)
- $\sqrt{3} \cdot x^2 - 1 = 0$ (a=$\sqrt{3}$, b=0, c=-1)

Now we have dealt with linear equations, we would now explore quadratic equations in the form $ax^2+bx+c = 0$. One method of solving quadratic equations is to try isolating the variable or factor one side. Let us try isolating the variable in our first example.

Example 1.04

Solve the following quadratic equation:

$$x^2 + 3 = 12.$$

SOLUTION We can subtract 3 from both sides then find the square root. This gives us

$$x^2 = 9$$

and so

$$x = \boxed{\pm 3}$$

Side note: the two answers to a quadratic equation are called *roots*.

Isolating the variable is quite easy in the above example, but not in this next one. However, you don't need to find x by isolating to solve the problem.

Example 1.05

If x is a number such that $x^2+x-1=0$, what is the value of $4x^2+4x+1$? *(Source: MATHCOUNTS)*

SOLUTION Notice that $4\cdot(x^2+x-1)=4x^2+4x-4$. Since $x^2+x-1=0$, $4\cdot(x^2+x-1)=4\cdot 0=0$. Therefore by adding 4 to both sides of $4x^2+4x-4=0$, we get

$$4x^2+4x=4.$$

Let's look back to the question. We are looking for $4x^2+4x+1$. We already know that $4x^2+4x=4$. So $4x^2+4x+1=4+1=\boxed{5}$.

Concept

Sometimes you don't need to solve for x to find what you're looking for. Instead, look for shortcuts to reduce whole expressions of variables into numbers. Make sure you read the problem so you know what you're looking for!

Our next strategy is to factor out at least 1 side of the equation.

Example 1.06

If there exists a number a such that any x in the equation $x^2+9x+20=(x+4)(x+a)$ will always make the equations true, then what is a?

SOLUTION There are two ways we can approach this problem. We see that one side is factored and the other equation is expanded out. To solve this equation, we could either expand both of them or factor them.

Solution 1: Expand both sides

We see that the left side of our equation is already expanded. So we only need to expand the other side. Expanding the right side gives us

$$x^2+9x+20=x^2+(4+a)x+4a.$$

Subtracting by x^2 on both sides gives us

$$9x + 20 = (4 + a)x + 4a.$$

Looking at this we see that $9 = 4 + a$, (make sure you see why). By subtracting 4 on both sides we get

$$a = \boxed{5}.$$

Solution 2: Factor both sides

On the right side of the equation we see that there is an $(x+4)$. Because we're trying to solve for a, let's try to factor $(x+4)$ on the left side as well. It turns out that $x^2 + 9x + 20 = (x + 4)(x + 5)$. Now we have

$$(x + 4)(x + 5) = (x + 4)(x + a)$$

By now what a should be is clear.

$$a = \boxed{5}.$$

Exercises

Answer the following questions.

1.3.1 Find all values of r such that $r^2 + 3r - 70 = 0$. *(Source: MATHCOUNTS)*

1.3.2 What is the sum of the values of x for which $x^2 - 13x + 40 = 0$? *(Source: MATHCOUNTS)*

1.3.3 Expand the product $(x + 2)(x - 7)$. *(Source: MATHCOUNTS)*

Challenge Problems

Answer the following questions.

1.3.4 The quadratic equation $x^2 + bx + c = 0$ has real roots 4 and -6. What is the value of $b + c$? *(Source: MATHCOUNTS)*

1.3.5 Prove that $x^2 + bx + c = (x + p)(x + q)$ where $p + q = b$ and $pq = c$. What if there is an a term in front of the x^2? Will your original answer work, and if not, what is the modified version? This is called Vieta's formula, and there are Vieta's rule for cubic equations. Note the method and result, you'll find it useful in later math contest questions.

1.3.6 Suppose that the number a satisfies the equation $4 = a + a^{-1}$. What is the value of $a^4 + a^{-4}$? *(Source: AMC 10)*

1.3.7 Solve $-x^3 + 3x^2 + 3x + 1 = 0$. Compare the left hand side expression with the expansion of $(x + 1)^3$, and try to exploit isolation. *(Source: AIME II 2020)*

1.4 Completing the Square

There are many methods to solve any quadratic equation. One of these methods is called **completing the square**.

Example 1.07

What value(s) of x satisfy the equation $x^2 - 2x - 1 = 0$?

SOLUTION Unfortunately, we cannot find a factorization to this expression. Here we will use a very old trick: We'll substitute $x = u + 1$. (You'll see where this comes from later.)

$$x^2 - 2x - 1 = (u+1)^2 - 2(u+1) - 1 = 0$$

$$u^2 + 2u + 1 - 2u - 2 - 1 = 0$$

Now, you can cancel the $2u$ to get

$$u^2 - 2 = 0$$

Hey, we can use isolation now!

$$u^2 = 2$$

We can think of two possible values for u, which are

$$u = \sqrt{2} \textit{ or } -\sqrt{2}$$

Now we substitute back in $x = u + 1$, or rather $u = x - 1$:

$$x - 1 = \sqrt{2} \textit{ or } -\sqrt{2}$$
$$x = 1 + \sqrt{2} \textit{ or } 1 - \sqrt{2}$$

The above method relies on the fact that the linear term (i.e. the term with u only in it), cancels when we substitute $x = u + 1$, so we can use substitution. How do we know that we need to substitute $x = u + 1$, instead of something like $x = u + 2$?

The trick is called 'completing the square'. The following question shows this trick in more detail.

Example 1.08

Solve the equation $x^2 + bx + c = 0$, where b and c are any constant number. Inspect the result, and does every value of b and c result in the above equation having one answer?

SOLUTION We want to substitute $x = u + k$ for some constant k so that the linear terms go away and we can use isolation.

$$x^2 + bx + c = (u + k)^2 + b(u + k) + c = 0$$

Simplify the middle expression:

$$u^2 + 2uk + k^2 + bu + bk + c = u^2 + u(2k + b) + (k^2 + bk + c) = 0$$

We want the linear term to go away, so the coefficient of the linear term must be equal to 0. Thus, $2k + b = 0$, or $k = -\frac{b}{2}$. Now substitute k for $-\frac{b}{2}$:

$$u^2 + u(2k + b) + (k^2 + bk + c) = u^2 + (-\frac{b^2}{4} + c) = 0$$

Nice, we have gotten rid of the linear term, so we can simply use isolation!

$$u^2 = \frac{b^2}{4} - c$$

$$u = \sqrt{\frac{b^2}{4} - c} \; or \; -\sqrt{\frac{b^2}{4} - c}$$

Finally, substitute back $u = x - k = x + \frac{b}{2}$:

$$u = (x - k) = (x + \frac{b}{2}) = \sqrt{\frac{b^2}{4} - c} \; or \; -\sqrt{\frac{b^2}{4} - c}$$

$$(x + \frac{b}{2}) = \sqrt{\frac{b^2}{4} - c} \; or \; -\sqrt{\frac{b^2}{4} - c}$$

$$x = -\frac{b}{2} + \sqrt{\frac{b^2}{4} - c} \; or \; -\frac{b}{2} - \sqrt{\frac{b^2}{4} - c}$$

Now we have a handy formula for a quadratic equation of the form $x^2 + bx + c$. Most of the time, people will leave out using u and use $x - \frac{b}{2}$ instead. We'll use our knowledge to try and get a $(x - \frac{b}{2})^2$ term and use isolation.

$$(x - \frac{b}{2})^2 = x^2 - bx + \frac{b^2}{4}$$

$$x^2 - bx = (x - \frac{b}{2})^2 - \frac{b^2}{4}$$

Substituting:

$$x^2 + bx + c = (x - \frac{b}{2})^2 - \frac{b^2}{4} + c = 0$$

$$(x - \frac{b}{2})^2 = \frac{b^2}{4} - c$$

Here there is a little problem. What if $\frac{b^2}{4} - c$ is negative? What if it were zero? We'll deal with that technicality later, but you can guess what might be going on.

$$x - \frac{b}{2} = \sqrt{\frac{b^2}{4} - c} \; or \; - \sqrt{\frac{b^2}{4} - c}$$

Finally,

$$x = \frac{b}{2} + \sqrt{\frac{b^2}{4} - c} \; or \; \frac{b}{2} - \sqrt{\frac{b^2}{4} - c}$$

Here, you should remember the trick of being able to 'complete the square'. This comes from spotting some $x^2 + bx$ terms and knowing that you can change it to $(x + \frac{b}{2})^2 - \frac{b^2}{4}$, so that you can remove the linear term and make use of the method of isolation, or for inspecting equations and expressions.

Use the above trick to answer the following questions:

Exercises

1.4.1 Inspect the equation $x^2 + 1 = 0$. Are there any real solutions? Why or why not? Try using isolation and see what went wrong. What does the formula say?

1.4.2 Try to use the method of completing the square to find a formula for the quadratic equation $ax^2 + bx + c$. This equation is the more common form seen in all high school mathematics textbooks, and you should practice using it with a few examples.

1.4.3 In the above derivation of the formula for the general quadratic equation $x^2 + bx + c$, we have met a technicality where we need to find a value x that satisfies

$$(x - \frac{b}{2})^2 = \frac{b^2}{4} - c$$

Check each case: when the right hand side is
i) > 0,
ii) < 0,
iii) $= 0$, what solutions for x are there? How many different solutions are there for x?

1.4.4 Let's say you have an equation $x^2 + bx + c = 0$, and you can write it in the form $x^2 + bx + c = (x - p)(x - q) = 0$. Show that p and q are roots of this quadratic equation. But we just found a way to calculate the roots of the equation Example 1.08 above. Substitute the formula for the roots into the factored form, and now you can factor any quadratic equation if you can calculate the roots! (More practice: is your answer consistent with Vieta's rule? Try it!)

Challenge Problems

1.4.5 Suppose you can choose whatever number x can be, and then you'll calculate the value of $x^2 + bx + c$ (b and c are fixed constants). Complete the square and remember that the square of any real number is always positive. What value of x will you get the maximum value? What about getting the minimum value?

1.4.6 From question 1.4.5, suppose that $a > 0$, and the minimum of $ax^2 + bx + c$ is greater than 0. When does that happen, and are there any solution(s) to the equation $ax^2 + bx + c = 0$? Why or why not? Find the analogue when $a < 0$, and when $a = 0$.

1.5 Quadratic Formula

There is a formula to calculate any root of a quadratic equation called the Quadratic Formula. If a variable x satisfies the equation $ax^2 + bx + c = 0$, then the possible values of x are:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If we add the the roots we get the sum of the roots as $(\frac{-b}{a})$ which we also get from Vieta's rule. What if we multiply them together? Does Vieta's rule hold also?

Now we will also get to know an important term from here known as the "discriminant".

Concept

The discriminant is the part which is under the square root in the quadratic formula, which is:

$$\boxed{b^2 - 4ac}$$

It also plays an important role in determining the nature of the roots.

You might have guessed that some quadratic equations have two distinct real roots, some have no real roots, and some only have one. The 'nature of the roots' is the name for what kind of roots a particular quadratic have.

The nature of the roots means either the roots are imaginary or real. If they are real then you can determine whether they are equal by looking at the discriminant of the equation.

There are three different cases: either the discriminant is less than 0, equal to 0, or more than 0. Let's go over these three cases in detail.

Case I: the discriminant is less than 0

The roots are imaginary. If you check your quadratic equation, the value inside the square root is negative, and your calculations will break down if you only look at real numbers. Another way to think is if you try completing the square, you'll notice that halfway you'll get

$$(x - \frac{b}{2a})^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$$

and the right hand side is obviously negative. But we cannot find a real number that has a negative square! So we conclude that there are no real answers to x. We'll see though later a 'number' that when squared gives a negative number. We call them 'imaginary' numbers. Ooh, spooky!

Case II: the discriminant is equal to zero

The roots are equal and real, meaning that there is only one real root. This is similar to when the discriminant is larger than zero.

Case III: the discriminant is more than 0

The roots are real. This means that there are two possible answers to the quadratic. Here there is no problem with the quadratic equation, or by completing the square.

Now, let's go over some problems to further develop our understanding of this formula.

Example 1.09

Determine the number of roots in each of the following cases:

a) $x^2 - 2x - 1 = 0$

b) $5x^2 + 8x + 20$

c) $16x^2 + 24x + 9$

SOLUTION a) We can find the number of roots in each equation by looking at the discriminant. For the first equation, the discriminant is:

$$(-2)^2 - 4 \cdot (-1) \cdot 1 = 8$$

Because $8 > 0$, there are $\boxed{\text{two}}$ roots to this equation.

b) Once again, let's look at the discriminant.

$$8^2 - 2 \cdot 5 \cdot 20 = -136$$

Because $-136 < 0$, there are $\boxed{\text{no roots.}}$

c) We repeat our process one more time.

$$24^2 - 4 \cdot 9 \cdot 16 = 0$$

When the discriminant is 0, we know that there is only $\boxed{\text{one root.}}$

Example 1.10

What are the roots of $x^2 + 5x + 4$?

SOLUTION We start off by defining a, b, and c in our equation. We see that $a = 1, b = 5$, and $c = 4$. Now, we can plug this into our equation.

$$\frac{-5 \pm \sqrt{5^2 - 4 \cdot 1 \cdot 4}}{2 \cdot 1} = \frac{-5 \pm \sqrt{9}}{2}$$

$$\frac{-5 \pm 3}{2} = -1, -4$$

Thus, our two roots are $\boxed{x = -1, -4.}$

Exercises

1.5.1 What is the sum of all real numbers x for which $|x^2 - 12x + 34| = 2$? *(Source: AMC 10A)* (Here $||$ means to take the inside value and make it positive. E.g. $|5| = 5$, $|-\frac{3}{2}| = \frac{3}{2}$. Think in terms of x and expressions of x's as numerical values)

1.5.2 Mathew accidentally made a mistake with his calculation using the quadratic formula. He is supposed to solve for x in $3x^2 = 3x + 216$. He tried to compute for one of the roots. Here is his computation:

$$\frac{-3 + \sqrt{(-3)^2 - 4 \cdot 3 \cdot 216}}{2 \cdot 3} = -431\frac{2}{3}$$

What are his mistakes?

Challenge Problems

1.5.3 Let $P(x) = x^2 - 3x - 7$, and let $Q(x)$ and $R(x)$ be two quadratic polynomials also with the coefficient of x^2 equal to 1. David computes each of the three sums $P + Q$, $P + R$, and $Q + R$ and is surprised to find that each pair of these sums has a common root, and these three common roots are distinct. If $Q(0) = 2$, then $R(0) = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$. *(Source: AIME 2020) (Hint: Label unknown constants that you think are important, then Vieta's rule again.)*

2 Graphing Equations

2.1 Introduction

Graphs can be used to express algebraic relations and geometrical properties.

This is called the Cartesian plane. The horizontal black line is the x-axis, and the vertical black line is the y-axis.

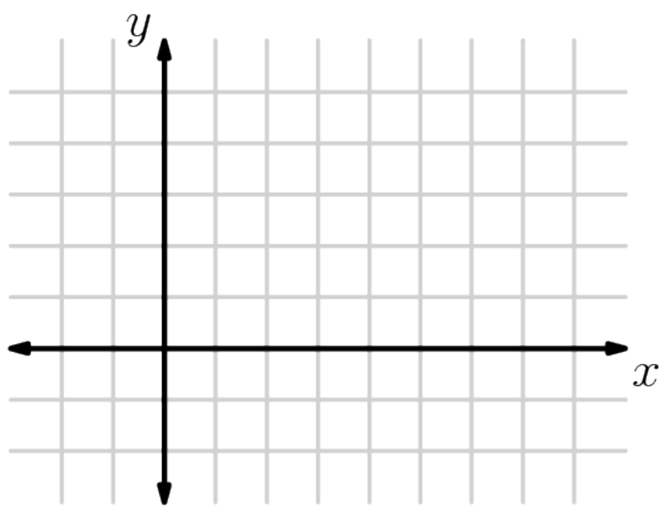


Figure 1: Graph of the Cartesian Plane

We can refer to any point in the plane with two numbers, by saying how much the point is to the right or to the left of the y-axis, and how much the point is to the top or bottom from the x-axis. If we want to write down the point that is x units to the right of the y-axis, and y units tall from the x-axis, we denote it like this:

$$(x, y)$$

If we want to say that a point is x units to the left of the y-axis instead of the right, we tag a minus sign $(-)$ onto the x value. Similarly, if you want to say that a point is y units below the x-axis, we will tag a minus sign into the y value. If a point is neither to the top nor to the bottom of the x-axis (in other words on the x-axis), then the y value will be 0, while a point on the y-axis will mean the x has a value of 0.

2.2 Graphing Linear Equations

Try the following exercises below before we begin graphing lines.

Exercises

2.2.1 Plot the point $(1, 7)$ in the Cartesian plane.

2.2.2 Plot the point $(-5, 2)$ in the Cartesian plane.

2.2.3 A frog sitting at the point $(1, 2)$ begins a sequence of jumps, where each jump is parallel to one of the coordinate axes and has length 1, and the direction of each jump (up, down, right, or left) is chosen independently at random. The sequence ends when the frog reaches a side of the square with vertices $(0, 0), (0, 4), (4, 4)$, and $(4, 0)$. What is the probability that the sequence of jumps ends on a vertical side of the square? *(Source: AMC 10A)*

Now, we will see another use of graphs. We will be dealing with mathematical input and output machines, sometimes known as functions. Here is an example: $y = 2x + 6$. It is conventionally accepted that x will be your input value and y is your output value. You let x be equal to the input, calculate the right hand side $2x + 6$, and your calculated value will be your output (represented by y here).

Say you input $x = 1$. Calculate $2x+6 = 2 \cdot 1+6 = 8$. You get that $y = 8$. Hey, you might think about $(x, y) = (1, 8)$. Now, go to your graph and find the point $(1, 8)$. This point will be on your graph, and you'll make a cross mark or dot on $(1, 8)$ to signify that $(1, 8)$ is on your graph. (The format of the other points on your graph will look like $(x, y) = (input, output)$. This is a very powerful way of representing input-outputs geometrically.)

You might realise that when you graph $y = 2x + 6$ you get a straight line. Mathematically, you *do* get a straight line (see exercises). A straight line graph has the equation of the form $y = mx + b$, where m and b are placeholders for constant numbers, x is your input, and y is your output. There are also other kinds of graphs you can make, such as $y = ax^2 + bx + c$, but these are not straight line graphs.

We will now see how to graph linear equations with this first example:

Example 2.01

If the line $y = 2x - 6$ intersects the line $x = 10$ at the point $(10, b)$, what is the value of b? *(Source: MATHCOUNTS)*

SOLUTION Here's a tip: If a point (x, y) is on the line $y = 2x - 6$, then x and y satisfies the equation $y = 2x - 6$. Similarly, if a point (x, y) is on the line $x = 10$, then x is simply 10, and y can be any number.[1]

So we have an intersection of the line $y = 2x + 6$ and $x = 10$. This means that the point satisfies both of the equations

$$y = 2x + 6 \textit{ and } x = 10$$

Now we have two equation with two variables, and we know how to solve them. Just substitute $x = 10$ into $y = 2x + 6 = 2 \cdot 10 + 6 = 26$, so $y = 26$. Now we know that $x = 10$ and $y = 26$.

The question says that the point $(10, b)$ is the intersection of the lines $y = 2x + 6$ and $x = 10$, but we know that $(10, 26)$ is also the intersection of the two lines. Well, there is only one intersection of the two lines, so they have to be the same point! $(10, b) = (10, 26)$, and so $b = \boxed{26}$.

Concept

The x axis can also be represented as $y = 0$ and the y axis can be represented as $x = 0$. Try to think as to how various horizontal or vertical lines can be expressed in such forms.

Another helpful concept to know is the slope of the graph.

[1] Here is one way of thinking about graphs. It is logically equivalent to the above statement, and it is a good exercise to think about why is that.

Concept

The slope of a line is a quantitative way of expressing how much the line goes up. It is equal to the difference of the y-coordinates over the difference of the x-coordinates, or the rise over the run. The rise is how much the line rose and the run is how much it went right. (Note that in mathematics, slope refers to a certain numerical value, not a geometrical description. But it does have important geometrical significance as you'll see later.)

Imagine you are going up a hill. You walked 5 meters in one direction forward and realised that you've went up by 1 meter. We say that the slope is

$$\frac{\text{(How much you've went up)}}{\text{(How much you've gone forward horizontally to flat ground)}} = \frac{1\text{ meter}}{5\text{ meters}} = \frac{1}{5}$$

.

Now imagine that you decided to walk down the hill on the other side, and you found that for every 3 meters that you went forward, you went down 2 meters. In this case, the slope is actually negative:

$$\frac{\text{(How much you've went up vertically)}}{\text{((How much you've gone forward horizontally to flat ground)}} = \frac{-2\text{ meters}}{3\text{ meters}} = -\frac{2}{3}$$

The $+$ or $-$ sign is very helpful to distinguish if you are going up and going down.

You might like to get a piece of graph paper to follow the below discussion. Say you start at some point (x_1, y_1) in the Cartesian plane and walked in a straight line towards another point (x_2, y_2). You have gone from x_1 units away from the y-axis to x_2 units from the y-axis, which means that you've went $x_2 - x_1$ units horizontally. Similarly, you have also gone from y_1 units away from the x-axis to y_2 units away from the x-axis, meaning that you've went $y_2 - y_1$ units vertically. Thus the slope is $\frac{\text{signed vertical distance}}{\text{signed horizontal distance}}$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

You can check that the above is true even if $x_2 < x_1$, if you are careful with your signs. Note that a vertical line has a slope of infinity, or undefined, because you will have gone some way up but no way right or left, and so when you try to calculate the slope you'll have to divide by 0 and you can't do that.

Now you can see the geometric significance of the slope, which is how 'steep' a graph is. Differential calculus is actually concerned with the 'steepness' of straight graphs, curved graphs and pointy graphs, and once you've taken calculus you'll see more insight into how slopes play an important role in increasing our understanding of graphs. (See exercises for a preliminary introduction into the study of slopes.)

Example 2.02

Plot the line $y = 5x + 4$ on the Cartesian plane.

(a) Find the slope of the line.

(b) Find the y-intercept of the line. (The y-intercept means the point where the graph 'cuts' through the y-axis. Sketch the graph and you'll see that there is indeed one point.)

Do you see anything interesting?

SOLUTION

(a) Imagine that you were to start somewhere on the graph, walk along it in the direction of increasing x and end somewhere else. Let's make things convenient and imagine you start at $(0, 4)$ and end at $(1, 9)$ (these points are on the graph). You thus have walked 1 unit to the right and went up 5 units, so the slope is 5. Or if you prefer the formula above:

$$\frac{9 - 4}{1 - 0} = \underline{\mathbf{5}}$$

(b) The y-intercept is when x is equal to 0 (why?). Putting it back into the equation we get $5 \cdot 0 + 4 = \underline{\mathbf{4}}$

Notice that our slope is the same as the coefficient of x. This isn't a coincidence! The coefficient of x is always equal to the slope. The same can be said for the y-intercept. The constant in our equation is always the y-intercept.

Concept

In the linear equation $y = mx+b$, m is the slope and b is the y-intercept.

For a good exercise: try proving the above. By doing problems, get a feel for slopes, graphs and their corresponding equations to build an intuitive understanding of them.

Let's use what we just learned in our next problem.

Example 2.03

Plot the line $y = -3x$ on the Cartesian plane. Now plot a line that is perpendicular to the one you just drew. What is the equation of that line?

SOLUTION Before we do anything let's graph the line perpendicular to the line $y = -3x$. Let's call this equation $y = mx + b$, where m and b are constants. We don't need to find x or y, because we are finding an input output machine that takes in an input x and gives out an output y.

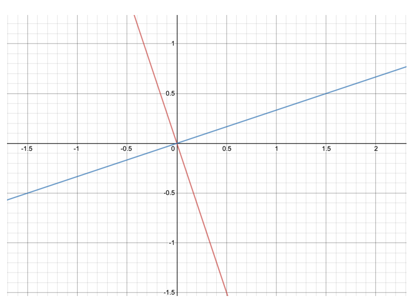


Figure 2: A graph of the line $y = -3x$ (in red) and another line perpendicular to it (in blue)

Using figure 2 to find two points on the blue line, we find that $(0, 0)$ and $(3, 1)$ are on the blue line. Now we just have to find the slope and y-intercept. The slope would be the m in the equation $y = mx + b$:

$$m = \frac{1-0}{3-0} = \frac{1}{3}$$

To find the y-intercept (the 'b') we want our input output machine to: input $x = 0$ and output $y = 0$ (because the point $(0, 0)$ is on the graph), and also input $x = 3$ and output $y = 1$ (because the point $(3, 1)$ is on our graph.) So we have

$$y = \frac{x}{3} + b$$
$$0 = \frac{0}{3} + b$$
$$b = 0$$

So our final equation is

$$\boxed{y = \frac{1}{3}x}$$

We can check that $(3, 1)$ is also on the line that we found above, so all is consistent.

Do you see anything interesting? The slopes of the two perpendicular lines when multiplied is equal to -1.

Concept

The slopes of two perpendicular lines when multiplied always equals -1.

Now we'll learn about the x-coordinate in our next and last example.

Example 2.04

What is the x-intercept for the line of the equation $y = 2x + 4$?

SOLUTION We need to graph the equation in order to see the x-intercept, which is when $y = 0$.

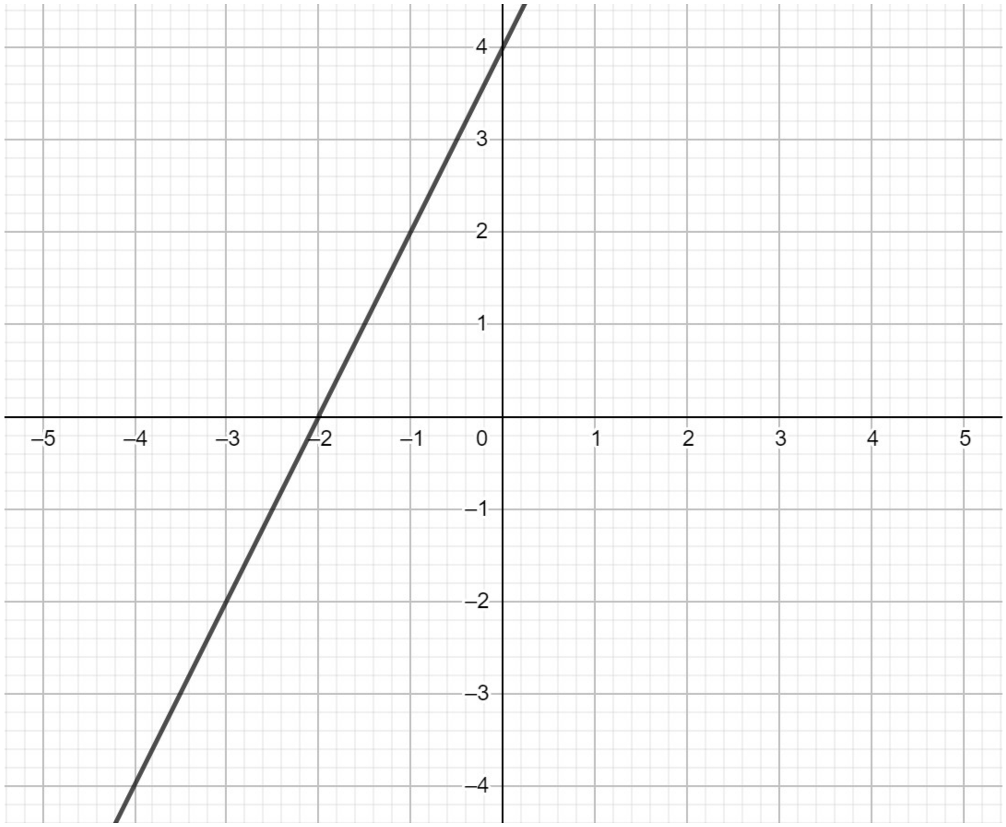


Figure 3: Graph of the equation $y = 2x + 4$

Looking at our graph, we see that $x = -2$ when $y = 0$. Let's put this back into our equation to check our answer:

$$2 \cdot -2 + 4 = 0$$

This shows that our answer is correct and the x-intercept is $\boxed{(-2, 0)}$.

Alternatively, you can solve this problem without graphing it. Intuitively, we know that the x-intercept of the graph is when $y = 0$, and the y-intercept of the graph is when $x = 0$. Thus, by plugging in $y = 0$ into our equation, we can solve for the x-intercept.

Exercises

2.2.4 On the same piece of graph paper, plot these points and join each pair with a line, then calculate their slope:

1. $(0, 0)$ and $(2, 1)$
2. $(0, 0)$ and $(2, 2)$
3. $(0, 0)$ and $(2, 4)$
4. $(0, 0)$ and $(2, -1)$
5. $(0, 0)$ and $(2, -2)$
6. $(0, 0)$ and $(2, -4)$

Make your own examples if you want instead, and make a general guess on how the value of the slopes affect the shape of the lines. Can you give any reasons for that?

2.2.5 What is the area of the triangle formed by the lines $y = 5$, $y = 1 + x$, and $y = 1 - x$? *(Source: AMC 8)*

2.2.6 The graphs of $2y+x+3 = 0$ and $3y+ax+2 = 0$ are perpendicular. Solve for a. *(Source: MATHCOUNTS)*

Challenge Problems

2.2.7 Say you have a slanting line, and you want to find the slope. How many distinct values can the slope of the slanting line have ? Conversely, say I have two lines with two different slopes. Can they have the same steepness on a graph?

2.2.8 Find a method of moving an arbitrary graph one unit to the right. How about moving the graph one unit upwards? What about rotating the graph by 90 degrees counterclockwise (and clockwise)? Try also to flip the graph across the x-axis and the y axis, and some other lines like the line $y = x$ and $x = 1$.

2.3 Graphing Quadratic Equations

We've already learned how to graph and read linear equations. Now we're going to move on to quadratic graphs, which are graphs with the equation of the form $y = ax^2 + bx + c$, similar to quadratic equations that we've seen before.

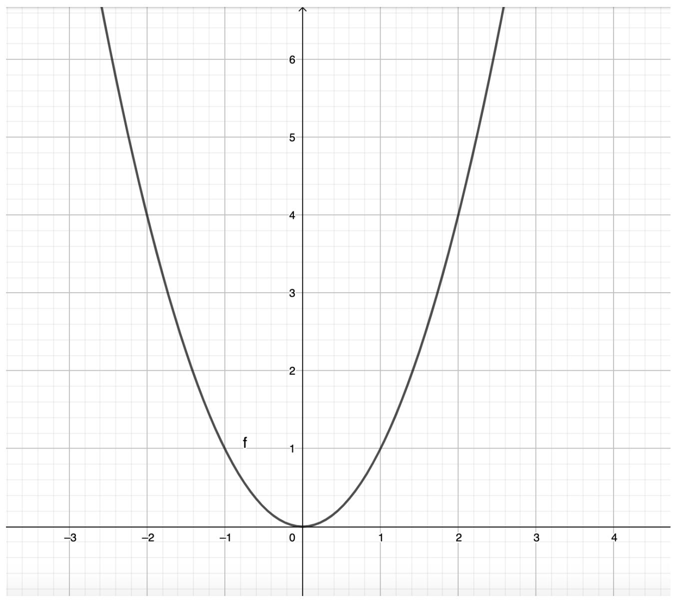


Figure 4: Graph of the equation $y = x^2$

Figure 4 is the graph of the equation $y = x^2$. Do you notice that it's a curve and not a line? This curve (and all graphs of the general quadratic equation $y = ax^2 + bx + c$) is called a parabola. Now lets try graphing some quadratics.

Exercises

2.3.1 Graph the equation $y = x^2 + 1$. What special features can you infer from the graph? Try to prove your guesses algebraically with the equation given. (Shape? Symmetry? Curliness? Max/Min value?)

This is just one of the infinitely graphs that can be graphed on the Cartesian plane. Let's look at some more examples.

Example 2.05

Find the vertex of the equation $y = 4x^2 + 3x + 9$.

SOLUTION First we need to know what the vertex of a parabola is.

Concept

The vertex of a parabola is the lowest or highest point of the parabola. Sometimes it is also called a stationary point or critical point.

Make a quick sketch, and notice that there is always a 'bump' in any quadratic equation. Then you might also notice that if you draw a pretty accurate quadratic graph, you can chop the graph in half with a vertical line at the peak or the trough of the bump, and you'll end up with two very symmetrical-looking parts. We'll see that it is an exact reflection in the exercises.

To see the vertex, we have to identify the parabola's lowest point.

We can plug this into a graphing calulator to see that the vertex is $\boxed{(-0.38, 8.44)}$. This is a method known as eyeballing a 'line of symmetry'. It should seem like the two halves are reflections of each other. At the vertex, the graph looks 'flat', so if you were to put a wooden plank onto the bottom side of the curve, the plank will be horizontal.

We can actually use graphs to analyse quadratic inequalities:

Example 2.06

Solve the inequality $0 < 4x^2 + 3x + 9$. (i.e. What values of x will satisfy that inequality?)

SOLUTION There are actually 3 ways of solving quadratic inequalities (completing the square, roots and using graphs), but here we will use the notion of quadratic graphs to understand them.

What quadratic graph will be useful here? Probably $y = 4x^2 + 3x + 9$. Let's draw that graph again and interpret it geometrically:

You can see that whatever the x value, your y value is always positive. This means that computing the value for $4x^2 + 3x + 9$ for any input x will always get you a positive number. Hence, the answer is $\boxed{\text{any input } x \text{ will do.}}$

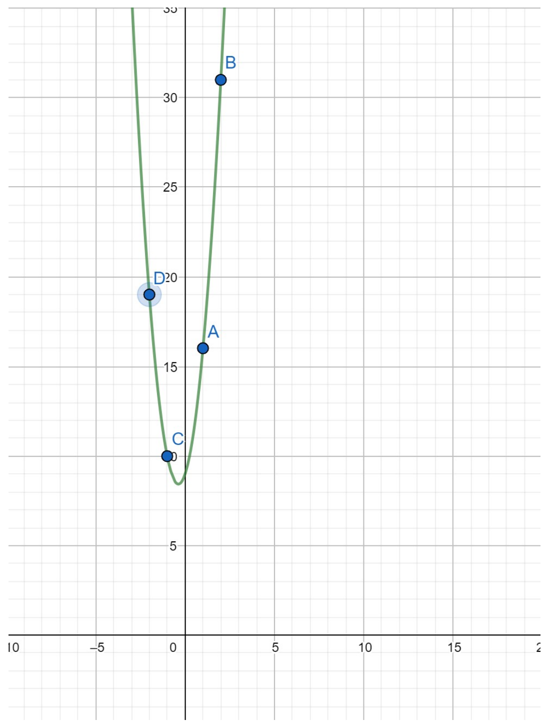


Figure 5: Graph of the equation $y = 4x^2 + 3x + 9$

Alternatively, we can try to find the roots of the equation. By plugging in the quadratic formula, we get:

$$\frac{-3 \pm \sqrt{3^2 - 4 \cdot 4 \cdot 9}}{2 \cdot 4}$$

Using what we learned about discriminants, we notice that $3^2 - 4 \cdot 4 \cdot 9 < 0$, so there are no roots to this equation.

What does this tell us? Because we know that there are no roots to this equation, we know that the graph will never touch the x-axis. Therefore, all of the points on the graph of $4x^2+3x+9$ must be either all above or all below the x-axis.

We can plug in any point to check this. By plugging in $x = 0$, we see that $4 \cdot 0^2 + 3 \cdot 0 + 9 > 0$. Thus, all of our points have a positive y-coordinate, so the answer is $\boxed{\text{any } x \text{ will satisfy this equation.}}$

Exercises

For each exercise, draw an important conclusion on graphs and quadratics, and create a general understanding and bag of tricks for approaching quadratic equations and graphs. Try to create a general formula or procedure for each question for the general formula $y = ax^2+bx+c$. You might even want to buy a notebook to write down your understanding, clever tricks and methods for future reference or to further strengthen your current understanding.

2.3.1 How is the general quadratic equation $ax^2 + bx + c = 0$ and the x-intercept(s) of the graph $y = ax^2 + bx + c$ related?

2.3.2 Completing the square is going to play a very important role in analyzing graphs. For the graph $y = 4x^2+3x+9$, first complete the square. Then notice that there is a bracket which is squared and in there an x term. Remember that x is an input value, what value(s) should you input for x to get the smallest value? What about getting the highest value? Can you now predict the vertex of that graph algebraically now?

Challenge Problems

2.3.4 Convince yourself that the graph of any polynomial is continuous and prove it. (A polynomial is a kind of algebraic expression with the form $a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1x + a_0$, where n is some positive integer and the a_i's are placeholders for constants.)

2.3.5 Apart from drawing quadratics, you can draw graphs of polynomials (e.g. $y = 3x^5 + 5x^3 - 2$), exponentials ($y = 2^x \cdot 3$), logarithms ($y = log_3x$), rational polynomials, and any formula that takes in an input x and spits out a real output y. Make your own examples and graph them, and see what kinds of patterns and rules you can extract from each kind of graph. More information can be found in general precalculus classes.

2.4 Graphing Inverses

Now that we have covered linear and quadratic functions, let's try graphing the **inverse** of a function.

First, what is the inverse of a function? The inverse of a function is the function that "undoes" the said function.

To understand this more thoroughly, let's try a few problems.

Example 2.07

Define a function $f(x)$ such that $f(x) = 2x + 1$. What is the function $g(x)$ such that $g(f(x)) = x$? That is, find the function $g(x)$ that changes the output of $f(x)$ into x.

SOLUTION Let's start by plugging $f(x) = 2x + 1$ into $g(f(x) = x$.

$$g(2x + 1) = x$$

Now, let's do a sneaky substitution. Define $u = 2x + 1$. Now, let's write x in terms of u.

$$u = 2x + 1$$
$$u - 1 = 2x$$
$$x = \frac{u-1}{2}$$

Thus, we know that $g(u) = \frac{u-1}{2}$, or $\boxed{g(x) = \dfrac{x-1}{2}}$ is the inverse function of $f(x)$.

Concept

If $g(x)$ is the inverse of $f(x)$, then $f(g(x)) = x$.

This is the same concept as "flipping" x and y. Our original equation is $y = 2x + 1$, so when we flip x and y we get $x = 2y + 1$.

We can simplify this equation to get

$$x = 2y + 1$$
$$x - 1 = 2y$$
$$y = \frac{x-1}{2}.$$

Once again, we find that the inverse of $g(x)$ is $\frac{x-1}{2}$.

Concept

Define $g(x)$ to be the inverse of $f(x)$. Then, if $f(x) = y$, $g(y) = x$.

While we know that $2x + 1$ has an inverse, can we say that for every single function? Let's try a few more functions in our next equation.

Example 2.08

Find the inverse of the following equations:

a) $f(x) = x^2$
b) $g(x) = x^3$

SOLUTION

a) At first glance, we notice that both -1 and 1 will output 1 in our equation. If this is the case, then what would $f'(1)$ equal?

Because $f'(1)$ has multiple solutions, this function $\boxed{\text{has no inverse.}}$

b) We "flip" x and y in our equation to get $x = y^3$. This is easy to simplify! We take the cube root of both sides to get that $y = \sqrt[3]{x}$.

Thus, the inverse of $g(x)$ is $\boxed{g'(x) = \sqrt[3]{x}.}$

Now that we know how to solve for the inverse of a function algebraically, let's try graphing them.

Example 2.09

In Example 2.07, we found that the inverse of $2x + 1$ was $\frac{x-1}{2}$. Graph both of these equations. What do you notice?

SOLUTION By now, you should be pretty familiar with graphing linear equations. Your graph should look something like this:

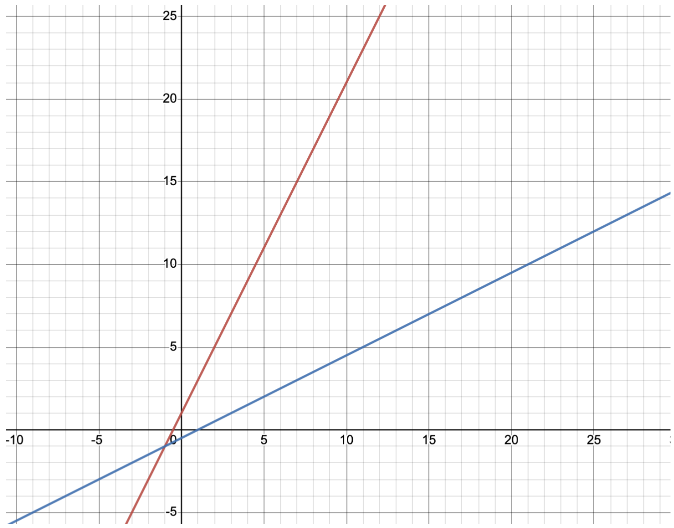


Figure 6: Graph of $2x + 1$ and $\frac{x-1}{2}$.

At first glance, we notice that the graph is very symmetric. In fact, it looks as if it is reflected across the line $y = x$. Let's graph this to check.

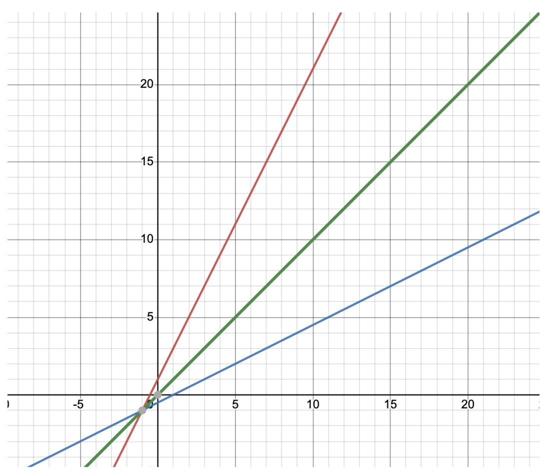


Figure 7: Graph of $2x + 1$ and $\frac{x-1}{2}$ with line $y = x$.

Yep, these two equations are definitely symmetric about $\boxed{y = x.}$

Concept

For any function $f(x)$ with an inverse, the graph of $f'(x)$ is the reflection of the graph of $f(x)$ about the line $y = x$.

If we look back at our solution in Example 2.07, this actually makes intuitive sense. When we "flip" x and y in our equation, we flip the x and y coordinates of the line. This is the same as reflecting our graph over the line $y = x$.

Now that you have a general idea concept of inverses and what they are, try doing these exercises.

Exercises

2.4.1 Find the inverse of $f(x) = 10x - 2$.

2.4.2 There exists functions $f(x)$ and $g(x)$ such that $f(g(x)) = x$. What is $g(f(x))$?

2.4.3 If $f(x) = \frac{3x}{2x-1}$, find $f^{-1}(x)$.

Challenge Problems

2.4.4 Find the inverse of $f(x) = x^2 + 2x - 6$.

2.4.5 There exists a function $g(x)$ such that $g(x) = 5x - 1$. If $g(x) = 3f^{-1}(x)$ and $f^{-1}(x)$ is the inverse of the function $f(x) = ax + b$, what is $a + b$?

2.4.6 From example 2.08, we saw that $f(x) = x^2$ has no inverse. Does this rule apply to all even functions (eg. all functions where all of the terms have even powers)?

3 Complex Numbers

3.1 Introduction

Is there a solution to $x^2 = -1$? Or equivalently, can we calculate $\sqrt{-1}$? You might think like this:

The square of a positive number is positive. The square of a negative number is positive. The square of 0 is 0. -1 does not have a square root!

Mathematicians may consider this fact. They say, "Well, we certainly cannot find an *exact* solution in the real numbers, but let's represent the solution with the letter i." You might object that i is not a number, but apparently it is one, the imaginary unit.

Right now, you can treat i as any normal variable, so all the normal rules of addition, multiplication, square rooting, indices and the rest of the algebraic rules applies (except one very annoying one which we'll deal with later). You'll learn why this is the case in more advanced classes and books, but for simplicity's sake just imagine i as a normal variable having the property that $i^2 = -1$.

3.2 Complex Numbers Arithmetic

Like real numbers, imaginary numbers also follow very similar arithmetic rules.

Example 3.01

(a) Add the complex numbers $3i$ and $2i$.

(b) Subtract the complex numbers $3i + 1$ and $2i$.

SOLUTION

(a) This is just like combining like terms in algebra. $3i + 2i = (3 + 2)i = \underline{\boldsymbol{5i}}$.

(b) This time we have added 1 to $3i$. We combine like terms again, just subtract $3i - 2i = i$ and add back the 1. We get $\underline{\boldsymbol{i + 1}}$.

If you want to multiply or divide complex numbers, you follow the normal rules of multiplying and dividing, as in this next example.

Example 3.02

(a) Multiply the numbers $3i$ and $2i$.

(b) Divide the numbers $3i$ and $2i$.

SOLUTION
(a) You're probably thinking that $3i \cdot 2i = 6i^2$. You're correct! However, we can simplify this equation even more. Remember that $i^2 = -1$. So

$$3i \cdot 2i = 6i^2 = 6 \cdot (-1) = \boxed{-6}$$

In the mean time, be careful with this manipulation:

$$i^2 = (\sqrt{-1})^2 = \sqrt{-1} \cdot \sqrt{-1} = \sqrt{-1 \cdot -1} = \sqrt{1} = 1$$

But hey, shouldn't $i^2 = -1$? That creates an annoying inconsistency, and the best way to avoid it is to avoid the above manipulation. For most practical purposes, replace $\sqrt{-1}$ with i as soon as you see one whenever you are manipulating complex numbers is enough to avoid the above, and all other operations are legal.

(b) We can write this as a fraction so it is easier to see what we need to cancel.

$$3i \div 2i = \frac{3i}{2i}.$$

We see that each term has an i in it, so we can just get rid of it. The answer is $\boxed{\frac{3}{2}}$.

Now let's try to understand the weird things about complex numbers when we see them in equations. Remember that we can manipulate i just like any normal variable, and treat the complex number like any algebraic expression.

Example 3.03

Find a generalized simplification of $(a + bi)(c + di)$.

SOLUTION We can multiply this out like a normal equation:

$$(a + bi)(c + di) = ac + adi + bdi + bdi^2$$

We know that $i^2 = -1$ from our definition of i, so we can simplify this to equal $\boxed{ac - bd + (ad + bd)i}$.

From this we can see that imaginary numbers follow many of the same rules as real numbers.

So far, we have covered addition, subtraction, and multiplication. Now let's cover division.

Example 3.04

Let $w = 6 - 8i$ and $z = 9 - 12i$. Solve for the following:

a) $w + z$

b) $w \cdot z$

c) $\star \frac{w}{z}$

SOLUTION

a) This one is relatively simple. We can split up the real and imaginary parts to solve:

$$\begin{aligned}(6 - 8i) + (9 - 12i) &= (6 + 9) + (-8 - 12)i \\ &= \boxed{15 - 20i}\end{aligned}$$

b) We can simplify out our equation by multiplying our equation out.

$$\begin{aligned}(6 - 8i)(9 - 12i) &= 6 \cdot 9 - 9 \cdot 8i - 6 \cdot 12i + 8 \cdot 12i^2 \\ &= 54 - 72i - 72i - 96 \\ &= \boxed{-42 - 144i}\end{aligned}$$

c) This one is a little tricky. Let's start by trying to factor w and z. We can factor w as $2(3 - 4i)$ and z as $3(3 - 4i)$.

Hey, they share a common factor! We can now easily divide this to get $\frac{2(3-4i)}{3(3-4i)} = \boxed{\dfrac{2}{3}}$.

Another property of an imaginary number is its **magnitude**.

Definition

The magnitude of a complex number in the form of $a + bi$ is equal to $\sqrt{a^2 + b^2}$.

Let's use this property in a few questions.

Example 3.04

What is the magnitude of $6 + 8i$? Does this number look familiar?

SOLUTION We plug in $6 + 8i$ into our equation to get $\sqrt{6^2 + 8^2} = \boxed{10}$.

Hey, that number looks familiar! 10 is the same value you would get if you were trying to find the distance from the point $(6, 8)$ to the origin.

In fact, the magnitude is the same thing as the distance to the origin of the **complex plane.** We'll go over this more later in section 3.4.

Example 3.05

What is $x\overline{x}$, if x is a complex number? How does this relate to the magnitude of x?

SOLUTION We start by denoting x as $a + bi$. Let's write this equation out.

$$\begin{aligned}(a + bi)(a - bi) &= a^2 + abi - abi - b^2i^2 \\ &= a^2 + b^2\end{aligned}$$

Thus, we get that $x\overline{x} = a^2 + b^2$. Now, how do we write $a^2 + b^2$ in terms of x? Well, we know that $\sqrt{a^2 + b^2} = |x|$, so we can denote $a^2 + b^2$ as $|x|^2$.

With this, we reach our conclusion that $\boxed{x\overline{x} = |x|^2}$

Concept

For some complex number x, $x\overline{x} = |x|^2$.

Exercises

3.2.1 Simplify $(4+i)(3-5i)$.

3.2.2 Simplify $(6-i)(4-2i)$

3.2.3 Simplify the sequence i^1,i^2,i^3,i^4,... Find a pattern. Then calculate the sequence $\frac{1}{i}$,$\frac{1}{i^2}$,$\frac{1}{i^3}$,... How is this pattern related to the first pattern, and can you give an explanation?

Challenge Problems

3.2.4 Simplify $(5+2i)^3$

3.2.5 Find as many solutions to the equation $(a+bi)^3 = 1$, where a and b are real numbers. (Hint: there are 3 solutions.)

3.2.6 You know that the solution to $x^2+1=0$ is i, but what about $x^2+4=0$? Do we have to invent another number j that is the solution to $x^2+4=0$? Luckily for us, $x=2i$ satisfies that equation (check for yourself). We don't actually need to invent anymore numbers, apparently. The complex numbers are enough to solve any equation that we throw to you.

For example, try solving the equation $x^2+2x+5=0$ by completing the square first, then guess at a possible value for $x+1$. (Is it i? $2i$? $-2i$?)

3.2.7 Remember your general quadratic formula $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. Imagine that the discriminant b^2-4ac is negative. What kinds of roots does your quadratic now have? Can you now factor the corresponding equation ax^2+bx+c in the complex numbers?

3.3 Conjugates of Complex Numbers

Just like how every real number has a reciprocal, every complex number has a **conjugate**.

Definition

The conjugate of a complex number $a + bi$ is $a - bi$ and denoted by $\overline{a + bi}$.

Let's explore this concept in more detail.

Example 3.05

What is the value of x where $(2 + 5i)(x - 5i)$ is real?

SOLUTION We can multiply out the equation to simplify:

$$(2 + 5i)(x - 5i) = (2x + 25) + (5x - 10)i$$

For this number to be real, we want the imaginary part to be equal to 0. We set $5x - 10$ to equal 0 to get that x must equal $\boxed{2}$.

Hey, $2 + 5i$ and $2 - 5i$ are conjugates! Now let's find a generalized formula for this.

Example 3.06

Find the generalized formula for $a + bi$ multiplied by its conjugate.

SOLUTION We know that the conjugate of $a+bi$ is $a-bi$, so we can multiply our equation out to simplify:

$$\begin{aligned}(a + bi)(a - bi) &= a^2 + abi - abi - b^2i^2 \\ &= a^2 - b^2(-1) \\ &= \boxed{a^2 + b^2}\end{aligned}$$

Thus, $a + bi$ multiplied by its conjugate is $a^2 + b^2$.

Concept

For a complex number z, $z\overline{z} = |z|^2$.

Let's try to prove a few more theorems.

Example 3.07

Prove that $\overline{\overline{z}} = z$ for some complex number z.

SOLUTION Let z be equal to $a + bi$. We know that the conjugate of $a + bi$ is equal to $a - bi$.

Because we are applying this formula twice, we now only need to find the conjugate of $a - bi$.

$$a - bi = a + (-b)i$$

Thus, the conjugate of $a + (-b)i$ is $a - (-b)i = \boxed{a + bi.}$

Concept

For some complex number z, $\overline{\overline{z}} = z$.

Because we can find the conjugate of a complex number through addition and subtraction, the following rules will also apply:

Theorems

$$\overline{x + y} = \overline{x} + \overline{y}$$
$$\overline{x - y} = \overline{x} - \overline{y}$$
$$\overline{x^n} = \overline{x}^n$$

As an exercise, you can try to prove these theorems. Let's try doing a few more problems.

Example 3.07

Prove that if a polynomial P has a root of a complex number z, then $\overline{z}$ is also a root of this polynomial.

SOLUTION Let $P(x)$ be a polynomial with complex coefficients. Suppose z is a root of $P(x)$, i.e., $P(z) = 0$. Then, by the definition of complex conjugate, $\overline{z}$ is the complex conjugate of z. We want to show that $\overline{z}$ is also a root of $P(x)$, i.e., $P(\overline{z}) = 0$.

Since $P(x)$ has complex coefficients, we can write $P(x)$ in the form:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are complex numbers.

Now, substitute $x = z$ into $P(x)$:

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0$$

Taking the complex conjugate of both sides of the equation:

$$\overline{P(z)} = \overline{0}$$

$$\overline{a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0} = 0$$

Using the properties of complex conjugates, we have:

$$\overline{a_n z^n} + \overline{a_{n-1}z^{n-1}} + \cdots + \overline{a_1 z} + \overline{a_0} = 0$$

Since the complex conjugate of a product is the product of the complex conjugates and the complex conjugate of a sum is the sum of the complex conjugates, we get:

$$a_n \overline{z^n} + a_{n-1}\overline{z^{n-1}} + \cdots + a_1 \overline{z} + a_0 = 0$$

But $\overline{z^n} = (\overline{z})^n$, $\overline{z^{n-1}} = (\overline{z})^{n-1}$, and so on. Substituting these back into the equation, we obtain:

$$a_n(\overline{z})^n + a_{n-1}(\overline{z})^{n-1} + \cdots + a_1\overline{z} + a_0 = 0$$

This is precisely $P(\overline{z}) = 0$, which means $\overline{z}$ is a root of $P(x)$. Thus, if z is a root of a polynomial $P(x)$ with complex coefficients, then $\boxed{\overline{z} \text{ is also a root of } P(x)}$.

Exercises

3.3.1 What is the value of x if the expression $(3x+5i)(6-2i)$ is real?

3.3.2 If $x^2 = 4+3i$, what is $x\overline{x}$?

3.3.3 If p and q are real number such that x^2+px+q has a root of $3+4i$, what is $p+q$?

Challenge Problems

3.3.4 Prove that the following equations:

$$\overline{x+y} = \overline{x}+\overline{y}$$
$$\overline{x-y} = \overline{x}-\overline{y}$$
$$\overline{x^n} = \overline{x}^n$$

3.3.5 Given that $2-5i$ is a root of $f(x) = x^3-7x^2+41x-87$, solve for the other two roots of $f(x)$.

3.4 The Complex Plane

Similar to the Cartesian plane, the Complex plane is used to graph complex numbers, where real numbers take the place of the x-axis and imaginary numbers are the y-axis. As you would expect, these axes are called the real and imaginary axis.

For example, $2-i$ would be equivalent to $(2,-1)$ on the Cartesian plane.

Let's try to solve some problems regarding the complex plane.

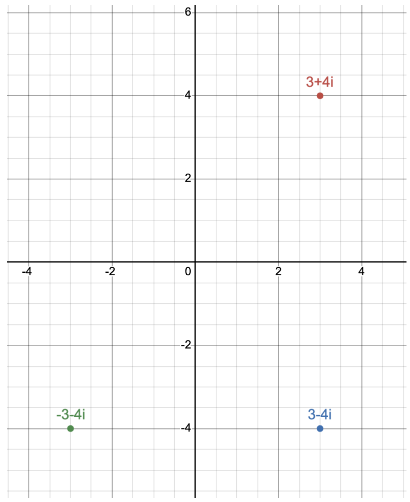
Example 3.08

Graph the following numbers on the Complex Plane.

$$3+4i, 3-4i, -3-4i$$

What are their relationships? What can you generalize about them?

SOLUTION We can graph the points as follows:



Now that we have these points graphed, let's try to find the relationship between these points.

Let's define x such that $x = 3 + 4i$. We know that $3 - 4i$ is $\overline{x}$ and $-3 - 4i$ is $-x$.

Using this, we can generalize a few rules about conjugates and negative complex numbers.

Concept

For any number x on the complex plane, $\overline{x}$ is the reflection of x over the real axis, and $-x$ is a 180 rotation of x.

These relationships should be intuitive to you, so don't worry too much about having them memorized.

Another interesting concept related to the complex plane is magnitude, which we discussed earlier. If you need a refresher, refer back to section 3.2.

Example 3.09

What is the magnitude of $3 + 4i$? Graph this point on the Complex Plane. What do you notice?

SOLUTION Using our equation from before, we find that

$$\sqrt{3^2 + 4^2} = \boxed{5}$$

Hey, that's the same as the distance from $3 + 4i$ to the origin!

We can generalize this to all complex numbers as follows:

Concept

The magnitude of a complex number is equivalent to its distance from the origin on the complex plane.

Let's use this in a few more problems.

Example 3.10

We know that the magnitude of $x = 3 + 4i$ and $y = 5 + 12i$ are 5 and 13, respectively, by the definition of magnitude. What do you notice about $|y - x|$?

We know that $y - x$ is equal to $2 + 8i$, so $|y - x| = \sqrt{2^2 + 8^2} = \boxed{\sqrt{70}}$.

Looking closely at this, we notice that the distance between x and y is also equal to $\sqrt{70}$. This is not a coincidence!

Concept

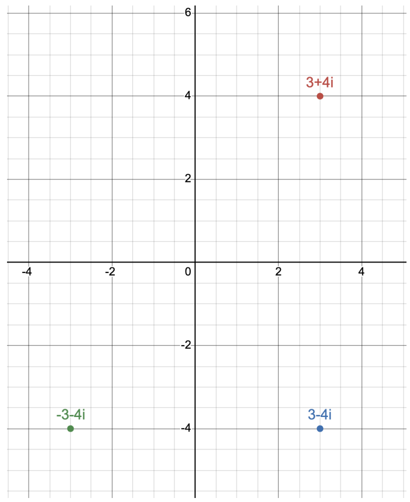
For every two complex numbers x and y, the distance between them can be defined as $|x - y|$.

As an exercise, you can try and prove this.

Let's try a few more problems.

Example 3.11

Graph the points $1 + 2i$ and $2 + 4i$. What do you notice? What can you generalize about this?



SOLUTION We can graph these points as follows:

We notice that the points $1 + 2i, 2 + 4i$, and 0 are collinear. Thus, we can assume that for any two complex numbers x and y such that $\frac{x}{y}$ is a real number, x and y are collinear with the origin.

Why is this?

Assume there exists a complex number w, and $z = aw$, where a is a real number. We can write w as $p + qi$ and z as $ap + aqi$. When graphing these on the Complex plane, we get that $w = (p, q)$ and $z = (ap, aq)$, which, along with the point $(0, 0)$, makes up the line $y = \frac{q}{p}x$.

Concept

For any two complex numbers x and y such that $\frac{x}{y}$ is a real number, x and y are collinear with the origin.

These equations should all be pretty intuitive to you, so don't worry too much about memorizing them. Come back to these equations every once in a while to refresh your brain!

Now, try to do some exercise problems on your own to get a better sense of the complex plane.

Exercises

3.4.1 Plot the following points on the complex plane:

a) $5 + 3i$
b) $2 - 3i$
c) $(5 + 3i)(2 - 3i)$

3.4.2 Find the distance between $2 + 3i$ and $7 - 9i$.

3.4.3 Find the magnitude of $\frac{2+7i}{7+2i}$.

Challenge Problems

3.4.4 Prove that the distance between two complex numbers w and z is $|w - z|$.

3.4.5 Find the area of the region of points x in the complex plane where $|x| = 4$. Can you generalize this formula?

3.4.6 Graph $|z - 4i| \leq 4$ on the complex plane.

What's Next?

Congratulations on making it all the way through Algebra (vol. 1)! The concepts in this book are not easy, so make sure you continuously come back to them to review.

The next three books in the series (Geometry, Number Theory, and Combinatorics), are coming out soon! While you are waiting for those to come out, visit our website at dragoncurvetutoring.org and sign up for one of our classes! We teach classes locally in the Montgomery County (MD) region, and we will be opening up asynchronous classes online soon! Make sure to join our mailing list for updates!

Once again, I'd like to thank Aritra12, Aops-g5-gethsemanea2, Rusczyk, MeHateMemes, RohanQV, and mathbw225 for their contributions to this book. This project wouldn't be here without you all.

In the meantime, here are a few suggestions to keep your math skills sharp:

1. **Practice, Practice, Practice:** Consistently work on problems to solidify your understanding. Consider using additional resources like Khan Academy, Brilliant.org, or Math Olympiad problems to challenge yourself further.

2. **Join a Math Club or Group:** Engaging with others who share your interest in mathematics can provide motivation and introduce you to new perspectives and problem-solving techniques.

3. **Explore Real-World Applications:** Look for ways mathematics is used in the real world. This could be through coding, finance, engineering, or even art and music. Understanding the practical applications can make learning more interesting and meaningful.

4. **Read Math-Related Books and Articles:** There are numerous books and articles written for a general audience that explore fascinating mathematical concepts.

5. **Attend Math Competitions and Workshops:** Participating in math competitions or attending workshops and seminars can enhance your skills and

provide exposure to advanced topics and new problem-solving strategies.

We hope these suggestions help you continue your mathematical journey with enthusiasm and curiosity. Stay tuned for more updates and happy learning!